

One-dimensional hydrogen atom with minimal length uncertainty and maximal momentum

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Abstract. We present exact energy eigenvalues and eigenfunctions of the one-dimensional hydrogen atom in the framework of the Generalized (Gravitational) Uncertainty Principle (GUP). This form of GUP is consistent with various theories of quantum gravity such as string theory, loop quantum gravity, black-hole physics, and doubly special relativity and implies a minimal length uncertainty and a maximal momentum. We show that the quantized energy spectrum exactly agrees with the semiclassical results.

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1. Introduction

In recent years, the investigation of the effects of the Generalized Uncertainty Principle (GUP) on various physical systems has attracted much attention and many authors have found exact or approximate solutions in both classical and quantum mechanical domains [1–4]. Indeed, because of the universality of this gravitational effect, it couples to all forms of matter and modifies the corresponding Hamiltonians in both non-relativistic and relativistic limits. Moreover, the existence of a finite lower bound to the possible resolution of length proportional to the Planck length $\ell_{Pl} = \sqrt{\frac{G\hbar}{c^3}} \approx 10^{-35}\text{m}$, where G is Newton's gravitational constant, naturally arises from various candidates of quantum gravity such as string theory [5–10], loop quantum gravity [11], and noncommutative spacetime [12–14]. Also the presence of a maximal momentum proportional to ℓ_{Pl}^{-1} is in agreement with Doubly Special Relativity (DSR) theories [15–17].

The problem of the hydrogen atom is studied in ordinary quantum mechanics and its well-known exact energy eigenvalues and eigenfunctions have already been obtained [18–22]. In the presence of the minimal length uncertainty, this problem is also studied exactly and perturbatively in Refs. [23–27]. Moreover, using the formally self-adjoint representation [28, 29], the quantization condition is completely determined in one-

dimension upon imposing the Hermiticity condition on the GUP-corrected Hamiltonian [30].

In this paper, we consider a recently proposed generalized uncertainty principle that implies both a minimal length uncertainty proportional to $\hbar\sqrt{\beta}$ and a maximal momentum proportional to $\frac{1}{\sqrt{\beta}}$ where β is the deformation parameter [31, 32]. The problems of the free particle, particle in box, harmonic oscillator, maximally localized states, black-body radiation, and cosmological constant have been studied in this framework [31, 32]. Here, we solve the problem of the one-dimensional hydrogen atom in this deformed quantum mechanics and find the exact energy eigenvalues and eigenfunctions in the momentum space. We show that imposing the Hermiticity condition on the Hamiltonian results in the self-adjointness of the Hamiltonian and the vanishing of the wave functions at the origin in coordinate space. We finally obtain the semiclassical energy spectrum that exactly agrees with the quantum mechanical results as well as with ordinary quantum mechanics and the modified quantum mechanics with just a minimal length.

2. Momentum space representation

Consider the following one-dimensional commutation relation (see Refs. [31, 32] for details):

$$[X, P] = \frac{i\hbar}{1 - \beta P^2}, \quad (1)$$

which to the first order of the GUP parameter agrees with the well-known GUP proposal by Kempf, Mangano and Mann [13]. To satisfy the above commutation relation, we write the position and momentum operators in the momentum space representation as

$$P\phi(p) = p\phi(p), \quad (2)$$

$$X\phi(p) = \frac{i\hbar}{1 - \beta p^2} \partial_p \phi(p). \quad (3)$$

Now the position operator is symmetric subject to the following scalar product

$$\langle \psi | \phi \rangle = \int_{-1/\sqrt{\beta}}^{+1/\sqrt{\beta}} dp (1 - \beta p^2) \psi^*(p) \phi(p), \quad (4)$$

and we have

$$\langle p | p' \rangle = \frac{\delta(p - p')}{1 - \beta p^2}. \quad (5)$$

In the framework of this generalized uncertainty principle, the absolutely smallest uncertainty in position is given by [31]

$$(\Delta X)_{min} = \frac{3\sqrt{3}}{4} \hbar \sqrt{\beta}. \quad (6)$$

and the maximal momentum is

$$P_{max} = \frac{1}{\sqrt{\beta}}. \quad (7)$$

Now consider the one-dimensional hydrogen atom eigenvalue problem

$$P^2\phi - \frac{\alpha}{X}\phi = E\phi, \quad (8)$$

where we set $\hbar = 1 = 2m$. In momentum space, the action of the inverse operator $1/X$ is expressed as

$$\frac{1}{X}\phi(p) = -i \int_{-\frac{1}{\sqrt{\beta}}}^p (1 - \beta q^2) \phi(q) dq + c, \quad \frac{-1}{\sqrt{\beta}} < p < \frac{+1}{\sqrt{\beta}}, \quad (9)$$

where, as we shall show, c is indeed a constant. The presence of c is due to the fact that the application of (9) with $c = 0$ only results in the trivial solution $\phi(p) = 0$ [27]. Moreover, in the absence of GUP, this constant corresponds to the derivative discontinuity of the eigenfunctions at the origin in the coordinate representation [19]. This definition implies

$$X \frac{1}{X} \phi = \phi, \quad (10)$$

$$\frac{1}{X} X \phi = \phi + c, \quad (11)$$

$$\left[X, \frac{1}{X} \right] \phi = -c. \quad (12)$$

In the same way we have $X^\dagger \phi = \frac{i}{1 - \beta p^2} \frac{\partial \phi}{\partial p}$ and the action of the adjoint of $1/X$ is given by

$$\left(\frac{1}{X} \right)^\dagger \phi(p) = -i \int_{-\frac{1}{\sqrt{\beta}}}^p (1 - \beta q^2) \phi(q) dq + c^*, \quad \frac{-1}{\sqrt{\beta}} < p < \frac{+1}{\sqrt{\beta}}, \quad (13)$$

Thus, we have

$$X^\dagger \left(\frac{1}{X} \right)^\dagger \phi = \phi, \quad (14)$$

$$\left(\frac{1}{X} \right)^\dagger X^\dagger \phi = \phi + c^*, \quad (15)$$

$$\left[X^\dagger, \left(\frac{1}{X} \right)^\dagger \right] \phi = -c^*. \quad (16)$$

At this point, we prove that X^{-1} is not a linear operator. In a basis which X (a linear operator) is diagonal, the formal operational relation $X \frac{1}{X} = 1$ (10) implies that if X^{-1} is a linear operator with a matrix representation, it is also diagonal. So we obtain

$$[X, X^{-1}] = 0, \quad (17)$$

which apparently contradicts Eq. (11). The same argument also applies for X^\dagger . Explicitly we have

$$\begin{aligned} \frac{1}{X} [\mu\phi(p) + \nu\varphi(p)] &= -i\mu \int_{-\frac{1}{\sqrt{\beta}}}^p (1 - \beta q^2) \phi(q) dq - i\nu \int_{-\frac{1}{\sqrt{\beta}}}^p (1 - \beta q^2) \varphi(q) dq + c, \\ &\neq \mu \frac{1}{X} \phi(p) + \nu \frac{1}{X} \varphi(p), \end{aligned} \quad (18)$$

and a similar relation for $\left(\frac{1}{X}\right)^\dagger$. Thus c is a constant not a linear operator, i.e., a linear functional.

The action of the position operator and its adjoint (9,13) leads to

$$\left[\frac{1}{X} - \left(\frac{1}{X}\right)^\dagger\right] \phi = 2\text{Im}[c]. \quad (19)$$

Since the momentum operator P is Hermitian, i.e. $P = P^\dagger$, the requirement of the Hermiticity for the Hamiltonian implies

$$\text{Im}[c] = 0. \quad (20)$$

Here, as we shall see, by imposing the Hermiticity condition, we completely determines the quantization condition that uniquely determines the energy spectrum.

The generalized Schrödinger equation in momentum space now reads

$$p^2 \phi(p) + i\alpha \int_{-\frac{1}{\sqrt{\beta}}}^p (1 - \beta q^2) \phi(q) dq - \alpha c = -\epsilon \phi(p), \quad (21)$$

where $\epsilon = -E$. If we differentiate this equation with respect to p we obtain

$$\phi'(p) + \frac{2p + i\alpha(1 - \beta p^2)}{p^2 + \epsilon} \phi(p) = 0. \quad (22)$$

The solution then reads

$$\phi(p) = \frac{\mathcal{A} e^{i\alpha\beta p}}{p^2 + \epsilon} \exp \left[-i \frac{\alpha(1 + \beta\epsilon)}{\sqrt{\epsilon}} \arctan \left(\frac{p}{\sqrt{\epsilon}} \right) \right]. \quad (23)$$

It can be expressed as

$$\phi(p) = \frac{\mathcal{A} e^{i\alpha\beta p}}{p^2 + \epsilon} \left(\frac{1 - ip/\sqrt{\epsilon}}{1 + ip/\sqrt{\epsilon}} \right)^{\frac{\alpha(1+\beta\epsilon)}{2\sqrt{\epsilon}}}, \quad (24)$$

where \mathcal{A} is the normalization coefficient. Substituting the above expression into the eigenvalue equation (21) leads to

$$c = \frac{1}{\alpha} \lim_{p \rightarrow -\frac{1}{\sqrt{\beta}}} (p^2 + \epsilon) \phi(p) = \mathcal{A} e^{-i\alpha\sqrt{\beta}} \left(\frac{\sqrt{\beta\epsilon} + i}{\sqrt{\beta\epsilon} - i} \right)^{\frac{\alpha(1+\beta\epsilon)}{2\sqrt{\epsilon}}}. \quad (25)$$

Therefore, the probability density in momentum space is

$$|\phi(p)|^2 = \frac{\mathcal{A}^2}{(p^2 + \epsilon)^2}, \quad (26)$$

and the normalization coefficient can be written as

$$\mathcal{A} = \frac{\epsilon^{3/4}}{\sqrt{(1 - \beta\epsilon) \text{arccot}(\sqrt{\beta\epsilon}) + \sqrt{\beta\epsilon}}}. \quad (27)$$

By imposing the Hermiticity condition (20) we find

$$\sin \left[\alpha \frac{1 + \beta\epsilon}{\sqrt{\epsilon}} \text{arccot}(\sqrt{\beta\epsilon}) - \alpha\sqrt{\beta} \right] = 0, \quad (28)$$

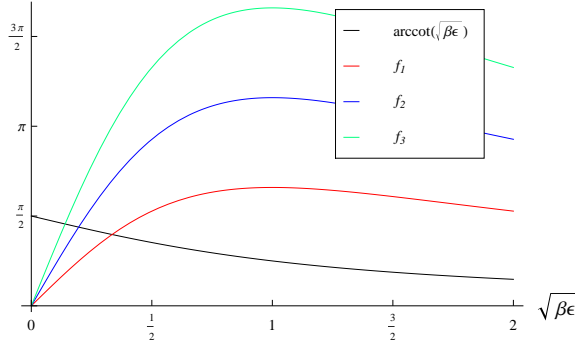


Figure 1. Schematic solutions of $\text{arccot}(x) = f_n(x)$, where $x = \sqrt{\beta\epsilon}$, $f_n(x) = \frac{1+n\pi/(\alpha\sqrt{\beta})}{x+1/x}$, and $\alpha = 1/\sqrt{\beta}$.

so the quantization condition reads

$$\frac{1 + \beta\epsilon}{\sqrt{\epsilon}} \text{arccot}(\sqrt{\beta\epsilon}) - \sqrt{\beta} = \frac{n\pi}{\alpha}, \quad n = 1, 2, \dots \quad (29)$$

It is straightforward to check that at the limit $\beta \rightarrow 0$ the above condition agrees with the non-deformed energy condition, i.e.,

$$\frac{\alpha}{2\sqrt{\epsilon}} = n, \quad n = 1, 2, \dots \quad (30)$$

Moreover, we have

$$\int_{-\frac{1}{\sqrt{\beta}}}^{+\frac{1}{\sqrt{\beta}}} (1 - \beta p^2) \phi(p) dp = \frac{2\mathcal{A}}{\alpha} \sin \left[\alpha \frac{1 + \beta\epsilon}{\sqrt{\epsilon}} \text{arccot}(\sqrt{\beta\epsilon}) - \alpha\sqrt{\beta} \right] = 0. \quad (31)$$

The quantization condition (29) can be rewritten as

$$\text{arccot}(x_n) = \left(1 + \frac{n\pi}{\alpha\sqrt{\beta}} \right) \left(x_n + \frac{1}{x_n} \right)^{-1}, \quad (32)$$

where $x_n = \sqrt{\beta\epsilon_n}$. It is now obvious that for each nonzero n there always exists a solution for Eq. (32). In figure 1, we have depicted the schematic solution for the above equation. In the presence of just a minimal length, the quantization condition is given by [30]

$$x_n = \frac{1}{2} \sqrt{1 + \frac{2\alpha\sqrt{\beta}}{n}} - \frac{1}{2}. \quad (33)$$

In Table 1, we have reported the first ten solutions of the quantization equations in ordinary quantum mechanics (30) and two GUP scenarios (32,33). As the table shows because of the maximal momentum all the energy levels which are proportional to $-x_n^2$ increase with respect to the presence of just a minimal length.

To check the self-adjointness of the Hamiltonian, since the momentum operator is obviously symmetric, we require that operator $1/X$ be a symmetric operator on the set of eigenfunctions

$$\left\langle \frac{1}{X} \varphi \middle| \phi \right\rangle = \left\langle \varphi \middle| \frac{1}{X} \phi \right\rangle, \quad (34)$$

Table 1. Solutions of the quantization equations in ordinary quantum mechanics and two GUP scenarios for $\alpha = 1/\sqrt{\beta}$.

n	x_n		
	Absence of GUP	Minimal Length	Minimal Length and Maximal Momentum
1	0.500000	0.366025	0.335027
2	0.250000	0.207107	0.196343
3	0.166667	0.145497	0.140027
4	0.125000	0.112372	0.109061
5	0.100000	0.091608	0.089388
6	0.083333	0.077350	0.075758
7	0.071429	0.066947	0.065749
8	0.062500	0.059017	0.058084
9	0.055556	0.052771	0.052023
10	0.050000	0.047723	0.047110

where ϕ and φ belong to the domain of the Hamiltonian and its adjoint, respectively. We can rewrite this condition using the explicit expression for operator $1/X$ (9) as

$$\begin{aligned}
& i \int_{-\frac{1}{\sqrt{\beta}}}^{+\frac{1}{\sqrt{\beta}}} (1 - \beta p^2) \phi(p) dp \int_{-\frac{1}{\sqrt{\beta}}}^p (1 - \beta q^2) \varphi^*(q) dq + c^* \int_{-\frac{1}{\sqrt{\beta}}}^{+\frac{1}{\sqrt{\beta}}} (1 - \beta p^2) \phi(p) dp \\
& = -i \int_{-\frac{1}{\sqrt{\beta}}}^{+\frac{1}{\sqrt{\beta}}} (1 - \beta p^2) \varphi^*(p) dp \int_{-\frac{1}{\sqrt{\beta}}}^p (1 - \beta q^2) \phi(q) dq + c \int_{-\frac{1}{\sqrt{\beta}}}^{+\frac{1}{\sqrt{\beta}}} (1 - \beta p^2) \varphi^*(p) dp. \quad (35)
\end{aligned}$$

Using the identity

$$\int_{-\frac{1}{\sqrt{\beta}}}^{+\frac{1}{\sqrt{\beta}}} f(p) dp \int_{-\frac{1}{\sqrt{\beta}}}^p g(q) dq = \int_{-\frac{1}{\sqrt{\beta}}}^{+\frac{1}{\sqrt{\beta}}} g(p) dp \left[\int_{-\frac{1}{\sqrt{\beta}}}^{+\frac{1}{\sqrt{\beta}}} f(q) dq - \int_{-\frac{1}{\sqrt{\beta}}}^p f(q) dq \right], \quad (36)$$

we obtain

$$\begin{aligned}
& i \int_{-\frac{1}{\sqrt{\beta}}}^{+\frac{1}{\sqrt{\beta}}} (1 - \beta p^2) \phi(p) dp \int_{-\frac{1}{\sqrt{\beta}}}^{+\frac{1}{\sqrt{\beta}}} (1 - \beta q^2) \varphi^*(q) dq \\
& + c^* \int_{-\frac{1}{\sqrt{\beta}}}^{+\frac{1}{\sqrt{\beta}}} (1 - \beta p^2) \phi(p) dp - c \int_{-\frac{1}{\sqrt{\beta}}}^{+\frac{1}{\sqrt{\beta}}} (1 - \beta p^2) \varphi^*(p) dp = 0. \quad (37)
\end{aligned}$$

Therefore, because of (31) we have

$$\int_{-\frac{1}{\sqrt{\beta}}}^{+\frac{1}{\sqrt{\beta}}} (1 - \beta p^2) \varphi^*(p) dp = 0. \quad (38)$$

These results show that the domains of H and H^\dagger coincide and the Hamiltonian is rendered a true self-adjoint operator, i.e., $H = H^\dagger$ and

$$\mathcal{D}(H) = \mathcal{D}(H^\dagger) = \left\{ \phi \in \mathcal{D}_{\max} \left(\frac{-1}{\sqrt{\beta}}, \frac{+1}{\sqrt{\beta}} \right); \right.$$

$$\int_{-\frac{1}{\sqrt{\beta}}}^{+\frac{1}{\sqrt{\beta}}} (1 - \beta p^2) \phi(p) dp = 0 \Big\}. \quad (39)$$

3. Coordinate space representation

The eigenfunctions of the position operator satisfy the following eigenvalue equation

$$X u_x(p) = x u_x(p), \quad (40)$$

where $u_x(p) = \langle p|x \rangle$. In momentum space we have

$$\frac{i}{1 - \beta p^2} \partial_p u_x(p) = x u_x(p). \quad (41)$$

This equation can be solved to obtain the position eigenvectors

$$u_x(p) = \mathcal{N} \exp \left[-ipx \left(1 - \frac{\beta}{3} p^2 \right) \right]. \quad (42)$$

The eigenfunctions are normalizable

$$1 = \mathcal{N} \mathcal{N}^* \int_{-1/\sqrt{\beta}}^{+1/\sqrt{\beta}} dp (1 - \beta p^2) = \frac{4\mathcal{N}\mathcal{N}^*}{3\sqrt{\beta}}, \quad (43)$$

and we finally obtain

$$u_x(p) = \frac{\sqrt{3\sqrt{\beta}}}{2} \exp \left[-ipx \left(1 - \frac{\beta}{3} p^2 \right) \right]. \quad (44)$$

Now using the completeness relation

$$\int_{-\frac{1}{\sqrt{\beta}}}^{+\frac{1}{\sqrt{\beta}}} dp (1 - \beta p^2) |p\rangle \langle p| = 1, \quad (45)$$

we find the wave function in coordinate space

$$\psi(x) = \frac{\sqrt{3\sqrt{\beta}}}{2} \int_{-\frac{1}{\sqrt{\beta}}}^{+\frac{1}{\sqrt{\beta}}} (1 - \beta p^2) e^{ipx(1 - \frac{\beta}{3} p^2)} \phi(p) dp. \quad (46)$$

Now Eq. (31) implies

$$\psi(x) \Big|_{x=0} = 0. \quad (47)$$

So the coordinate space wave functions vanish at the origin, i.e., they obey the Dirichlet boundary condition.

4. Single-valuedness criteria

Alternatively, one may use the requirement of single-valuedness of eigenfunctions (24) to find the quantization condition which leads to

$$\frac{\alpha(1 + \beta\epsilon)}{2\sqrt{\epsilon}} = m, \quad m = 1, 2, \dots \quad (48)$$

However, the eigenfunctions obeying quantization condition (48) do not satisfy (31) and (47) and the Hermiticity condition (20) therefore fails. Comparison between the two quantization conditions (29) and (48) shows that

$$m = \frac{n\pi + \alpha\sqrt{\beta}}{2 \operatorname{arccot}(\sqrt{\beta\epsilon})}, \quad (49)$$

$$= n + \left(2n + \frac{\alpha}{\sqrt{\epsilon}}\right) \frac{\sqrt{\beta\epsilon}}{\pi} + 2 \left(2n + \frac{\alpha}{\sqrt{\epsilon}}\right) \frac{\beta\epsilon}{\pi^2} + \dots \quad (50)$$

So the single-valuedness criteria of the eigenfunctions ($m \in \text{integers}$) is only valid at the limit $\beta \rightarrow 0$, i.e., the absence of GUP. However, since for $\beta = 0$ all eigenfunctions satisfy

$$\int_{-\infty}^{+\infty} \phi(p) \, dp = 0 = \psi(0), \quad (51)$$

regardless of their energy, the single-valuedness condition is still valid at this limit.

Note that in ordinary quantum mechanics, the wave function associated with the particle must be single valued of its argument. Because, when two values are found, it means that the particle exists in two different places which is impossible when we consider particles as point-like objects. However, in the presence of the minimal uncertainty relation, the assumption of the point-like particles is no longer valid and the failure of the single-valuedness criteria is not problematic in this framework due to the fuzziness of space. Moreover, as Eq. (49) shows, this condition is only slightly broken in practise (small values of β) which agrees with the smallness of the uncertainty in position measurement (6). It is worth mentioning that the failure of the single-valuedness condition is well-known in the framework of the generalized uncertainty principle. For instance, as it is shown in the seminal paper by Kempf *et al.*, even the first GUP solutions such as the eigenfunctions of the position operator and the maximal localization states are not single valued in the presence of just the minimal length [13].

5. Semiclassical solutions

The semiclassical energy spectrum is given by the Bohr-Sommerfeld quantization condition

$$\oint p \, dx = 2n\pi, \quad n = 1, 2, \dots \quad (52)$$

The corresponding classical Hamiltonian to the hydrogen atom problem is

$$H(x, p) = p^2 - \frac{\alpha(1 - \beta p^2)}{x}, \quad (53)$$

where we used

$$X = \frac{1}{1 - \beta p^2} x, \quad x = i \frac{d}{dp}, \quad (54)$$

and neglected the ordering problem in classical domain. Since the Hamiltonian is conserved, i.e. $H(x, p) = E = -\epsilon$, we can express p as a function of x , namely,

$$p = \left(\frac{\alpha - \epsilon x}{x + \alpha\beta} \right)^{1/2}. \quad (55)$$

When the particle leaves the origin in positive direction, x changes from 0 to α/ϵ . So $\oint p \, dx = 2 \int_0^{\alpha/\epsilon} p \, dx$ and for the negative energy bound states we find

$$\begin{aligned} 2n\pi &= 2 \int_0^{\alpha/\epsilon} \left(\frac{\alpha - \epsilon x}{x + \alpha\beta} \right)^{1/2} dx \\ &= \frac{2\alpha(1 + \beta\epsilon)}{\sqrt{\epsilon}} \operatorname{arccot} \left(\sqrt{\beta\epsilon} \right) - 2\alpha\sqrt{\beta}, \end{aligned} \quad (56)$$

which exactly agrees with Eq. (29). The validity of the semiclassical approximation for this modified quantum mechanics is also discussed in [32].

6. Conclusions

In this paper, we have considered the problem of the one-dimensional hydrogen atom in the presence of both a minimal length uncertainty and a maximal momentum and found exact energy eigenvalues and eigenfunctions. By imposing the Hermiticity condition on the Hamiltonian, the quantization condition is uniquely determined and the Hamiltonian is rendered self-adjoint. We showed that the single-valuedness condition is only valid for the zero deformation parameter and the coordinate space wave functions vanish at the origin. Moreover, similar to the case where there is just a minimal length [30], the semiclassical energy spectrum exactly coincides with the quantum mechanical results.

References

- [1] S. Hossenfelder, arXiv:1203.6191.
- [2] P. Pedram, Europhys. Lett. **89** (2010) 50008.
- [3] P. Pedram and K. Nozari K. Europhys. Lett. **92** (2010) 50013.
- [4] P. Pedram, K. Nozari, and S.H. Taheri JHEP **1103** (2011) 093.
- [5] G. Veneziano, Europhys. Lett. **2** (1986) 199.
- [6] E. Witten, Phys. Today **49** (1996) 24.
- [7] D. Amati, M. Ciafaloni, G. Veneziano, Phys. Lett. B **216** (1989) 41.
- [8] D. Amati, M. Ciafaloni, G. Veneziano, Nucl. Phys. B **347** (1990) 550.
- [9] D. Amati, M. Ciafaloni, G. Veneziano, Nucl. Phys. B **403** (1993) 707.
- [10] K. Konishi, G. Paffuti, P. Provero, Phys. Lett. B **234** (1990) 276.
- [11] L.J. Garay, Int. J. Mod. Phys. A **10** (1995) 145.
- [12] M. Maggiore, Phys. Lett. B **319** (1993) 83.
- [13] A. Kempf, G. Mangano, R.B. Mann, Phys. Rev. D **52** (1995) 1108.
- [14] A. Kempf, G. Mangano, Phys. Rev. D **55** (1997) 7909.
- [15] J. Magueijo and L. Smolin, Phys. Rev. Lett. **88** (2002) 190403, arXiv:hep-th/0112090.
- [16] J. Magueijo and L. Smolin, Phys. Rev. D **71** (2005) 026010, arXiv:hep-th/0401087.
- [17] J.L. Cortes and J. Gamboa, Phys. Rev. D **71** (2005) 065015, arXiv:hep-th/0405285.
- [18] J.A. Reyes and M. del Castillo-Mussot, J. Phys. A **32** (1999) 2017.
- [19] Y. Ran, L. Xue, S. Hu and R-K. Su, J. Phys. A **33** (2000) 9265.
- [20] A.N. Gordeyev and S.C. Chhajlany, J. Phys. A **30** (1997) 6893.
- [21] I. Tsutsui, T. Fulop and T. Cheon, J. Phys. A **36** (2003) 275.
- [22] H.N. Nunez Yopez, C.A. Vargas and A.L.S. Brito, Eur. J. Phys. **8** (1987) 189.
- [23] R. Akhoury and Y.-P. Yao, Phys. Lett. B **572** (2003) 37.

- [24] D. Bouaziz, N. Ferkous, Phys. Rev. A **82** (2010) 022105.
- [25] F. Brau, J. Phys. A **32** (1999) 7691.
- [26] S. Benczik, L.N. Chang, D. Minic and T. Takeuchi, Phys. Rev. A **72** (2005) 012104.
- [27] T.V. Fityo, I.O. Vakarchuk and V.M. Tkachuk, J. Phys. A **39** (2006) 2143.
- [28] P. Pedram, Phys. Rev. D **85** (2012) 024016, arXiv:1112.2327.
- [29] P. Pedram, Phys. Lett. B **710** (2012) 478.
- [30] P. Pedram, J. Phys. A **45** (2012) 505304, arXiv:1203.5478.
- [31] P. Pedram, Phys. Lett. B **714** (2012) 317, arXiv:1110.2999.
- [32] P. Pedram, Phys. Lett. B **718** (2012) 638, arXiv:1210.5334.